3.1. Existence of closed geodesics. Let (M,g) be a compact Riemannian manifold and $c_0\colon S^1 \to M$ a continuous closed curve. Show that in the family of all continuous and piecewise C^1 curves $c\colon S^1 \to M$ which are homotopic to c_0, there is a shortest one. Prove that this is a geodesic.

Proof. Let us first prove that c_0 is homotopic to a piecewise C^1-curve c_1. To this aim, we split c_0 into finitely many paths $\gamma_i\colon [0,1] \to M$ such that $\gamma_i(1) = \gamma_{i+1}(0)$, $\gamma_n(1) = \gamma_1(0)$ and γ_i is contained in a charts $\{(\varphi_i, U_i)\}_{i=1}^n$ with U_i simply connected. Then γ_i is homotopic (relative to the endpoints) to a C^1-curve $\tilde{\gamma}_i$ and by connecting the $\tilde{\gamma}_i$'s we get a piecewise C^1-curve c_1 which is homotopic to c_0. Then c_1 has finite length $L(c_1)$.

Let $L := \inf_c L(c)$ be the infimum over all curves $c\colon S^1 \to M$ that are piecewise C^1 and homotopic to c_0 and consider a minimizing sequence, i.e. a sequence $(c_n\colon S^1 \to M)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} L(c_n) = L$.

We may assume that the curves $c_n\colon [0,1] \to M$ are parametrized proportionally to arclength, i.e. $L(c_n|_{[a,b]}) = |b-a| \cdot L(c_n)$. As M is compact, there is some $r > 0$ such that for all $p \in M$ it holds that for all $q, q' \in B(p, 3r)$ there is a unique geodesic from q to q' in $B(p,3r)$ and $B(p,3r)$ is simply connected.

Fix some $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{r}{L}$ and define $t_k := \frac{k}{N}$ for $k = 0, \dots, N$. Consider now the sequences $(c_n(t_k))_{n\in\mathbb{N}}$. By compactness of M, we may assume (by possibly passing to subsequences) that $c_n(t_k) \to p_k$ for each $k = 0, \dots, N$. We have

$$d(p_k, p_{k+1}) \le \limsup_{n\to\infty} d(c_n(t_k), c_n(t_{k+1})) \le \limsup_{n\to\infty} \tfrac{1}{N} L(c_n) < r$$

and therefore we can define a continuous, piece-wise C^1-curve $c\colon [0,1] \to M$ by concatenating the unique geodesics between p_k and p_{k+1}. For the length of c we have

$$L(c) = \sum_{k=0}^{N-1} L\left(c|_{\left[\frac{k}{N}, \frac{k+1}{N}\right]} \right) = \sum_{k=0}^{N-1} d(p_k, p_{k+1}) \le N \limsup_{n\to\infty} \tfrac{1}{N} L(c_n) = L.$$

It remains to prove that c is homotopic to c_0. Observe that for n large enough, we have $c(\left[\frac{k}{N}, \frac{k+1}{N}\right]), c_n(\left[\frac{k}{N}, \frac{k+1}{N}\right]) \subset B(p_k, 3r)$ and since $B(p_k, 3r)$ is simply connected there is a homotopy from $c_n|_{\left[\frac{k}{N}, \frac{k+1}{N}\right]}$ to $c|_{\left[\frac{k}{N}, \frac{k+1}{N}\right]}$ with the endpoints following the unique geodesics from $c_n(t_k)$ to p_k and from $c_n(t_{k+1})$ to p_{k+1}, respectively. Combining these homotopies, we get a homotopy from c_n to c.

Observe that c is locally length minimizing and hence a geodesic. $\square$

3.2. Homogeneous Riemannian manifolds. Let (M, g) be a *homogeneous Riemannian manifold*, i.e. the isometry group of M acts transitively on M. Prove that M is geodesically complete.

Solution. Fix some $p \in M$ and choose $\epsilon > 0$ such that geodesics through p exist on $B(p, 2\epsilon)$. Then for all $q \in M$, there exists an isometry $\varphi \in \mathrm{Isom}(M)$ such that $\varphi(q) = p$ and hence geodesics also exist on $B(q, 2\epsilon)$.

Let $\gamma\colon I \to M$ be a geodesic with $I \subset \mathbb{R}$ maximal and which is parametrized by arclength. For $t \in I$, we have $(t - \epsilon, t + \epsilon) \subset I$ by the above and therefore $I = \mathbb{R}$. $\square$

Note: By the Theorem of Hopf-Rinow this implies that M is complete.

3.3. Metric and Riemannian isometries. Let (M, g) and $(\bar{M}, \bar{g})$ be two connected Riemannian manifolds with induced distance functions d and $\bar{d}$, respectively. Further, let $f\colon (M, d) \to (\bar{M}, \bar{d})$ be an isometry of metric spaces, i.e. f is surjective and for all $p, p' \in M$ we have $\bar{d}(f(p), f(p')) = d(p, p')$.

(a) Prove that for every geodesic γ in M, $\bar{\gamma} := f \circ \gamma$ is a geodesic in M.

(b) Let $p \in M$. Define $F\colon TM_p \to T\bar{M}_{f(p)}$ with

$$F(X) := \frac{d}{dt}\bigg|_{t=0} f \circ \gamma_X(t),$$

where γ_X is the geodesic with $\gamma_X(0) = p$ and $\dot{\gamma}(0) = X$. Show that F is surjective and satisfies $F(cX) = cF(X)$ for all $X \in TM_p$ and $c \in \mathbb{R}$.

(c) Conclude that F is an isometry by proving $\|F(X)\| = \|X\|$.

(d) Prove that F is linear and conclude that f is smooth in a neighborhood of p.

(e) Prove that f is a diffeomorphism for which $f^*\bar{g} = g$ holds.

Solution. (a) As the property of being a geodesic is local, we may assume that both $\gamma\colon [0, L] \to M$ and $f \circ \gamma\colon [0, L] \to \bar{M}$ are contained in an open set $U \subset M$ and $\bar{U} \subset \bar{M}$, respectively, such that points in U and $\bar{U}$ are connected by a unique geodesic in U or $\bar{U}$. Then there is a unique geodesic β from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$. We claim that $\bar{\gamma}$ and β coincide.

In the following all geodesics are parametrized by arclength. For $t \in [0, L]$ there are geodesics β_1 from $\bar{\gamma}(0)$ to $\bar{\gamma}(t)$ and β_2 from $\bar{\gamma}(t)$ to $\bar{\gamma}(L)$. Concatenating β_1 and β_2, we get some piece-wise C^1-curve from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$ with length

$$\begin{aligned} L(\beta_1\beta_2) &= L(\beta_1) + L(\beta_2) \\ &= \bar{d}(\bar{\gamma}(0), \bar{\gamma}(t)) + \bar{d}(\bar{\gamma}(t), \bar{\gamma}(L)) \\ &= d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(L)) \\ &= d(\gamma(0), \gamma(L)) = \bar{d}(\bar{\gamma}(0), \bar{\gamma}(L)) = L(\beta). \end{aligned}$$

Hence, by uniqueness of the geodesic from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$, $\beta_1\beta_2$ and β coincide, i.e. $\bar{\gamma}(t) = \beta(t)$.

(b) Observe that f is bijective and its inverse f^{-1} is also is an isometry of metric spaces.

First, we prove that F is surjective. Let $Y \in T\bar{M}_{f(p)}$ and $\bar{\gamma}$ the geodesic through $f(p)$ with $\dot{\bar{\gamma}}(0) = Y$. Then $Y = F(X)$ for $X := \frac{d}{dt}\Big|_{t=0} f^{-1} \circ \bar{\gamma}(t)$.

From $\gamma_{cX}(t) = \gamma_X(ct)$ it follows that

$$F(cX) = \frac{d}{dt}\Big|_{t=0} f \circ \gamma_X(ct) = cF(X).$$

(c) For $\epsilon > 0$ small enough, we have that $\gamma_X(\epsilon)$ and $f \circ \gamma_X(\epsilon)$ are contained in a normal neighborhood of p and $f(p)$, respectively. Hence we get

$$\epsilon\|X\| = d(p, \gamma_X(\epsilon)) = \bar{d}(f(p), f \circ \gamma_X(\epsilon)) = \epsilon\|F(X)\|.$$

By the formula

$$2g_p(X, Y) = \|X\|^2 + \|Y\|^2 - \|X - Y\|^2$$

we conclude that $g_p(X, Y) = g_{f(p)}(F(X), F(Y))$.

(d) For all $X, Y, Z \in TM_p$ and $c \in \mathbb{R}$, we have

$$\begin{aligned} \bar{g}_{f(p)}(F(X + cY), F(Z)) &= g_p(X + cY, Z) \\ &= g_p(X, Z) + cg_p(Y, Z) \\ &= \bar{g}_{f(p)}(F(X), F(Z)) + c\bar{g}_{f(p)}(F(Y), F(Z)) \\ &= \bar{g}_{f(p)}(F(X) + cF(Y), F(Z)) \end{aligned}$$

Hence F is linear and therefore smooth.

If V_p is a neighborhood of $0 \in TM_p$ such that $\exp_p|_{V_p} \colon V_p \to U_p$ is a diffeomorphism, then we have

$$f|_{U_p} = \exp_{f(p)} \circ F \circ (\exp_p|_{V_p})^{-1}.$$

Hence f is smooth as well.

(e) The argument above works for all $p \in M$ and also for f^{-1}. Hence f is a diffeomorphism. Furthermore, we have

$$df_p = d(\exp_{f(p)} \circ F \circ \exp_p^{-1}) = F$$

and thus

$$f^*\bar{g}_p(X_p, Y_p) = \bar{g}_{f(p)}(df_p(X_p), df_p(Y_p)) = \bar{g}_{f(p)}(F(X_p), F(Y_p)) = g_p(X_p, Y_p),$$

for all $X, Y \in TM$. $\square$